Unconditioned extrema

Consider a company producing two goods A and B under conditions of perfect competition. Let p_1 and p_2 be the unit prices of each good respectively and let q_1 and q_2 be the production levels. If the cost function of the company is given by $C = 2q_1^2 + q_1 q_2 + 2q_2^2$, maximize the total profit for $p_1 = 12$ and $p_2 = 18$ and verify with second order conditions that it is a maximum.

Solution

$$C = 2q_1^2 + q_1q_2 + 2q_2^2$$
$$p_1 = 12$$
$$p_2 = 18$$

We know that:

$$B = R - C = (p_1 \cdot q_1 + p_2 \cdot q_2) - (2q_1^2 + q_1q_2 + 2q_2^2)$$

Substitute:

$$B = (12q_1 + 18q_2) - (2q_1^2 + q_1q_2 + 2q_2^2)$$

Finally, our function to optimize is:

$$B = 12q_1 + 18q_2 - 2q_1^2 - q_1q_2 - 2q_2^2$$

FOC (First Order Conditions):

$$B'_{q_1} = 12 - 4q_1 - q_2 = 0$$
$$B'_{q_2} = 18 - 4q_2 - q_1 = 0$$

Solve as a system of two equations using the substitution method:

$$12 - 4q_1 = q_2$$
$$18 - 4q_2 - q_1 = 0 \implies 18 - 4(12 - 4q_1) - q_1 = 0$$
$$18 - 48 + 16q_1 - q_1 = 0$$
$$-30 + 15q_1 = 0$$
$$q_1 = 2$$

Substitute the found q_1 into the other equation:

$$12 - 4 \cdot (2) = q_2 \implies 4 = q_2$$

We find our critical point: $P = (2; 4)$ Now calculate the second order conditions (SOC):

$$B''_{q_1q_1} = -4$$
$$B''_{q_1q_2} = B''_{q_2q_1} = -1 \quad \text{(By Schwarz's Theorem)}$$
$$B''_{q_2q_2} = -4$$

Construct the Hessian:

$$\mathbf{H} = \begin{bmatrix} B''_{q_1q_1} & B''_{q_1q_2} \\ B''_{q_1q_2} & B''_{q_2q_2} \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix}$$

Calculate the determinant:

$$\mathbf{H} = (-4)(-4) - (-1)(-1) = 16 - 1 = 15$$

Since $\mathbf{H} = 15 > 0$, we can confirm that there is a relative extremum.
To evaluate if it's a maximum or minimum, observe the sign of the second derivative $B''_{q_1q_1}$:

$$B''_{q_1q_1} = -4 < 0$$

Therefore, there is a maximum at $P = (2; 4)$ and the total profit value is:
Substitute into the objective function:

$$B = 12q_1 + 18q_2 - 2q_1^2 - q_1q_2 - 2q_2^2$$
$$B = 12 \cdot 2 + 18 \cdot 4 - 2 \cdot 2^2 - 2 \cdot 4 - 2 \cdot 4^2$$
$$B_{\text{total}} = 48$$

Answer: There is a maximum at $P = (2; 4)$ and the total profit value is 48.